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Gaze pattern: Investigation of strategies for dealing with indirectly proportional function graphs in a thermodynamic context

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Abstract

The ability to distinguish mathematical functions is crucial for understanding complex concepts in mathematics and physics. Differentiating graphs of indirectly proportional functions from other power functions poses significant challenges for many students, which is reflected in their strategies. This study examined the gaze behavior of 80 tenth-grade students when identifying isothermal changes of state in *p-V* diagrams. Using gaze data and retrospective interviews, two primary strategies were identified: a focus on value pairs, where pressure and volume values were compared, and a focus on the graph of function, where the curve's trajectory was analyzed. Students who focused on value pairs exhibited mostly horizontal and vertical gaze movements, while those concentrating on the graph of function demonstrated diagonal saccades. However, many students focusing on the graph of function paid attention to irrelevant aspects, such as the position of the curve, which led to confusion between isothermal and adiabatic changes of state. The results provide insights essential for developing adaptive teaching methods that are aligned with learners' specific strategies and eye movement patterns to improve their understanding of thermodynamic function graphs.

Die Unterscheidung mathematischer Funktionen ist entscheidend für das Verständnis komplexer Konzepte in Mathematik und Physik. Besonders die Differenzierung von Graphen indirekt proportionaler und anderer Potenzfunktionen stellt viele Lernende vor Herausforderungen, die sich in den Strategien der Lernenden widerspiegeln. Diese Studie untersuchte das Blickverhalten von 80 Zehntklässler*innen bei der Identifikation isothermer Zustandsänderungen in *p-V*-Diagrammen. Mithilfe von Eye-Tracking und retrospektiven Interviews wurden zwei Strategien identifiziert: der Fokus auf Wertepaare, bei dem Druck- und Volumenwerte verglichen werden, und der Fokus auf den Funktionsgraphen, bei dem der Kurvenverlauf analysiert wird. Lernende, die sich auf Wertepaare konzentrierten, zeigten vorwiegend horizontale und vertikale Blickbewegungen, während der Fokus auf den Funktionsgraphen mit diagonalen Blicksprüngen einherging. Viele Lernende fokussierten sich beim Betrachten des Funktionsgraphen jedoch auf irrelevante Aspekte, wie die Lage der Kurve, was zu Verwechslungen zwischen isothermen und adiabatischen Zustandsänderungen führte. Die Ergebnisse liefern wichtige Hinweise für die Entwicklung adaptiver Lehrmethoden, die an die spezifischen Strategien und Blickbewegungsmuster der Lernenden angepasst sind, um das Verständnis thermodynamischer Funktionsgraphen zu verbessern.

*Corresponding author: fehlinger@physik.rwth-aachen.de0 This article was originally submitted in German.

1. Introduction

Current educational research is increasingly concerned with the question of how digital learning methods can be effectively integrated into teaching in order to better support individual learning processes. A central goal is to investigate how learners use different forms of representation to successfully process complex information. Function graphs play a central role in STEM subjects, as they support learners in understanding functional relationships.

A typical example of this are indirectly proportional functions in thermodynamics, which describe isothermal state changes in p-V diagrams. They play a central role in both mathematics and physics lessons in secondary education.

However, learners often have difficulty grasping the meaning of such function graphs [1-3]. In the field of thermodynamics, this is particularly evident in the fact that learners have difficulty correctly distinguishing between isothermal and adiabatic state changes in a p-V diagram [2, 3]. This is also evident in the chosen processing strategies, which are not always effective when it comes to understanding the functional relationship, as exemplified in this article [1]. The problem here is that teachers face the challenge of accurately assessing their students' learning obstacles [4].

Eye tracking data analyses offer valuable insights into learners' strategies and can contribute to the optimization of teaching methods. In this study, they are used together with verbal data to investigate how learners recognize isothermal state changes and distinguish them from adiabatic state changes.

2. Isothermal change of state

In a reversible, isothermal state change, the temperature of the system remains constant, while the pressure is inversely proportional to the volume.
The product of pressure and volume remains constant. In the p-V diagram, such an isothermal state change is represented as a right-angled hyperbola (see Fig. 1A).

In reversible adiabatic state changes, there is no heat exchange with the environment because the system is thermally insulated [5]. In the p-V diagram, such adiabatic state changes are represented as power functions that show no indirect proportionality between pressure and volume (see Fig. 1B).

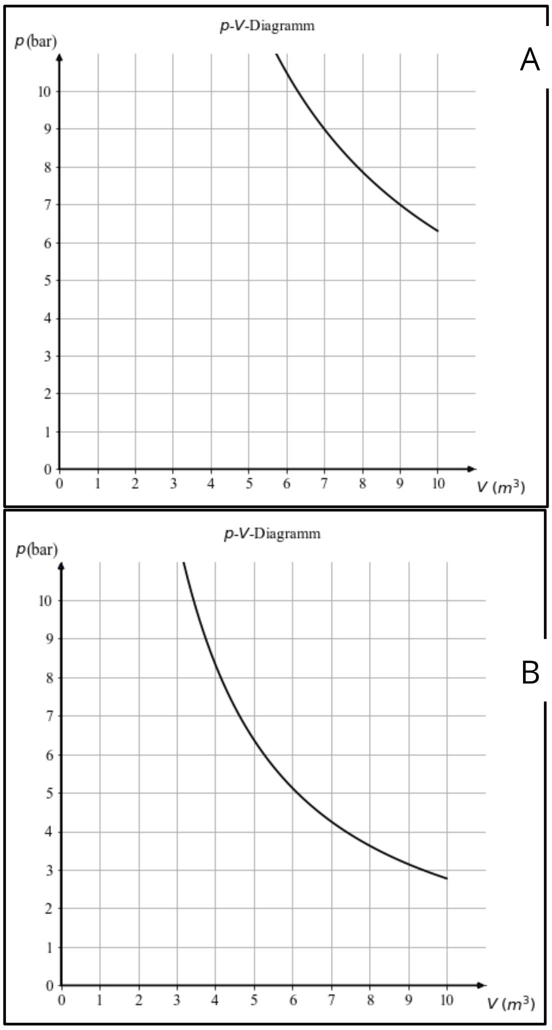

Fig. 1: Representation of an isothermal state change (A) and an adiabatic state change (B) in the p-V diagram. Item T1 from the empirical study published in PRPER [1].

Physical knowledge alone is not sufficient to interpret thermodynamic state changes. Functional thinking, which also includes mathematical understanding, is also required.

This includes, in particular, knowledge of proportional relationships, the course and interpretation of function graphs, and the relationship between variables such as pressure and volume.

Pinker [6] postulates that the understanding required to interpret a function graph is stored in the memory as a diagram scheme. Pinker [6] describes this diagram scheme as a knowledge structure that comprises information about operators that enable relevant data to be extracted from a function graph. Both essential elements of perception and processes of working and long-term memory play a decisive role in this [6].

3. Investigation of strategies

A deep understanding of the cognitive processes that learners use when working on tasks is crucial for optimizing teaching and learning processes. Eye tracking and retrospective think-aloud protocols (RLDP) are effective methods for uncovering the strategies used to solve tasks. These methods allow conclusions to be drawn about the underlying cognitive processes [7-11].

In eye tracking for the analysis of processing strategies, saccade directions are important in addition to transitions and fixations [7, 8]. Transitions show changes between fixed areas, while saccades represent rapid eye movements between fixation points. The direction of the saccades provides clues as to how texts or diagrams are read and interpreted [8].

Triangulation with RLDP enables further optimization of the analysis of processing strategies [9] by having participants reflect on their actions after completing a task [10].

In a study by Thomaneck et al. [11], gaze data and verbal data were used to examine strategies for analyzing changes in graphs in real-world contexts. This allowed two main strategies to be identified.

The first was the analysis of the entire function graph curve and the section-by-section examination based on specific value pairs. The verbal data corresponded well with the eye-tracking data and also provided valuable insights into the approaches used in the situational context [10].

This underscores the importance of eye tracking and supporting interviews to stimulate RLDP.

4. Objective

Based on the findings of Thomaneck et al. [11], this study aims to classify the processing strategies of learners in identifying graphs of indirectly proportional functions in a thermodynamic context and to analyze the saccade directions in the respective processing strategies. The goal is to analyze the processing strategies used when dealing with indirectly proportional function graphs and to relate them to the saccade directions, as these play an important role in the investigation of cognitive processes. This raises the following research question: *How can learners' processing strategies for identifying indirectly proportional function graphs in a thermodynamic context be classified, and what characteristic saccade directions can be assigned to these strategies?*

By combining eye-tracking data and verbal utterances, a deeper understanding of cognitive processes is to be gained.

5. Methodological approach

A total of 80 tenth-grade students from secondary schools in a medium-sized German city participated in the study (45 female, 34 male, 1 no information provided). The average age was 15.47 years (*SD* = 1.66); one person did not provide their age. The study was divided into three parts (see Fig. 2). In the first step, the students independently reviewed thermodynamic state changes in order to reactivate their knowledge from 8th grade using a handout.

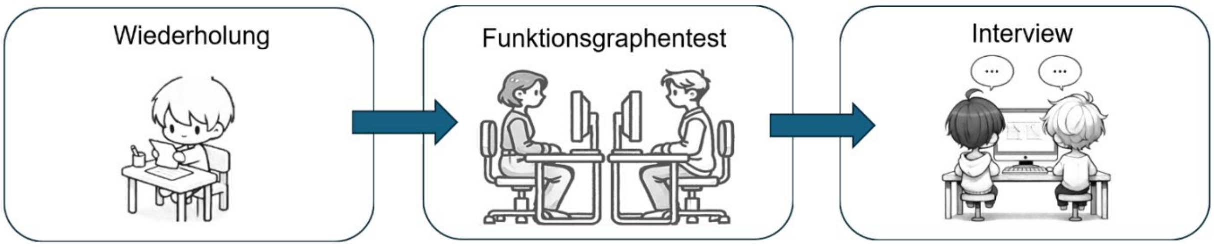

Fig. 2: Procedure of the study to investigate eye movement patterns during the identification of isothermal state changes in the p-V diagram. The image was created using ChatGPT-4o

A function graph test (see section 6) was then performed while simultaneously recording gaze data using a stationary eye tracker (Tobii Pro Fusion).Afterwards, the learners were asked questions about their approaches in order to gain a more detailed understanding of their processing strategies and to validate the eye tracking data. To assist them, the participants were shown a recording of their eye movements in the form of an eye path during the interview in order to aid their memory. A key advantage of eye tracks is that they map the processing steps step by step, enabling a detailed reconstruction of cognitive processes. Further details can be found in the study by Fehlinger et al. [1].

6. Function graph test

The study is based on a test instrument with three items in a thermodynamic context, which was developed to assess the understanding of function graphs (Cronbach's alpha = 0.75). For the present analysis, a representative item was selected that provides insight into the processing strategies of the learners. A more detailed evaluation of all items is documented in Fehlinger et al. [1].

The respective items of the test instrument were designed as multiple-choice tasks in which a distinction had to be made between isothermal and adiabatic state changes; the analyzed sample item represents a characteristic feature of this task format (see Figure 4). The goal was to correctly identify the isothermal state change.

Participants were asked to control their gaze behavior as precisely as possible so that an outsider could understand how the task was solved based on the recorded gaze paths. To make this easier, they were shown a recording of their own gaze paths from a previous task at the beginning of the study. In this way, the participants were to develop an understanding of what typical gaze paths look like and how their own gaze behavior can be represented by these gaze paths.

The test was conducted digitally, while the learners' eye movements were recorded using the Tobii Pro Fusion eye tracker. The scenario is shown in detail in Figure 3. The study leader and the participant each sat in front of a screen. The participant worked on the tasks on a screen with an eye tracker, while the study director followed the process in real time on another screen using the gaze paths.

This made it possible to identify and document initial anomalies in the processing in order to derive specific questions for the subsequent interview using a standardized questionnaire. Detailed descriptions of the methodological approach can be found in the article by Fehlinger et al. [1].

7. Evaluation method

For the evaluation of the sub-goal defined above, one item was used as an example, as it is particularly representative of the processing strategies and cognitive processes to be investigated.

The categorization of processing strategies when dealing with the graph of the indirectly proportional function was based on the gaze data.

To this end, the corresponding diagram was divided into relevant areas such as axes, origin, labels, arguments, function values, and the function graph itself, known as AOIs (Areas of Interest) (see Fig. 4).

The relevant areas were determined based on the representation of fixations in the heat maps (see Fig. 5).

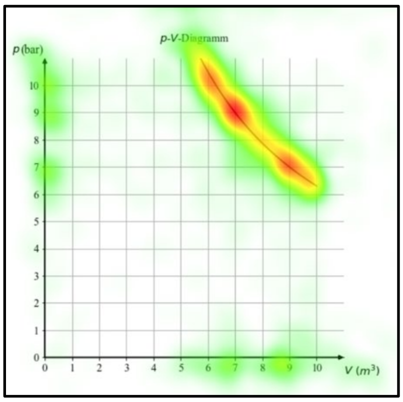

Fig. 5: Heat map showing the cumulative total fixation time over the entire processing time of 80 learners. Red = high, green = medium, yellow = low fixation duration. Item T1 from the empirical study published in PRPER [1].

To categorize the strategies, the number of transitions, fixations, and saccade directions

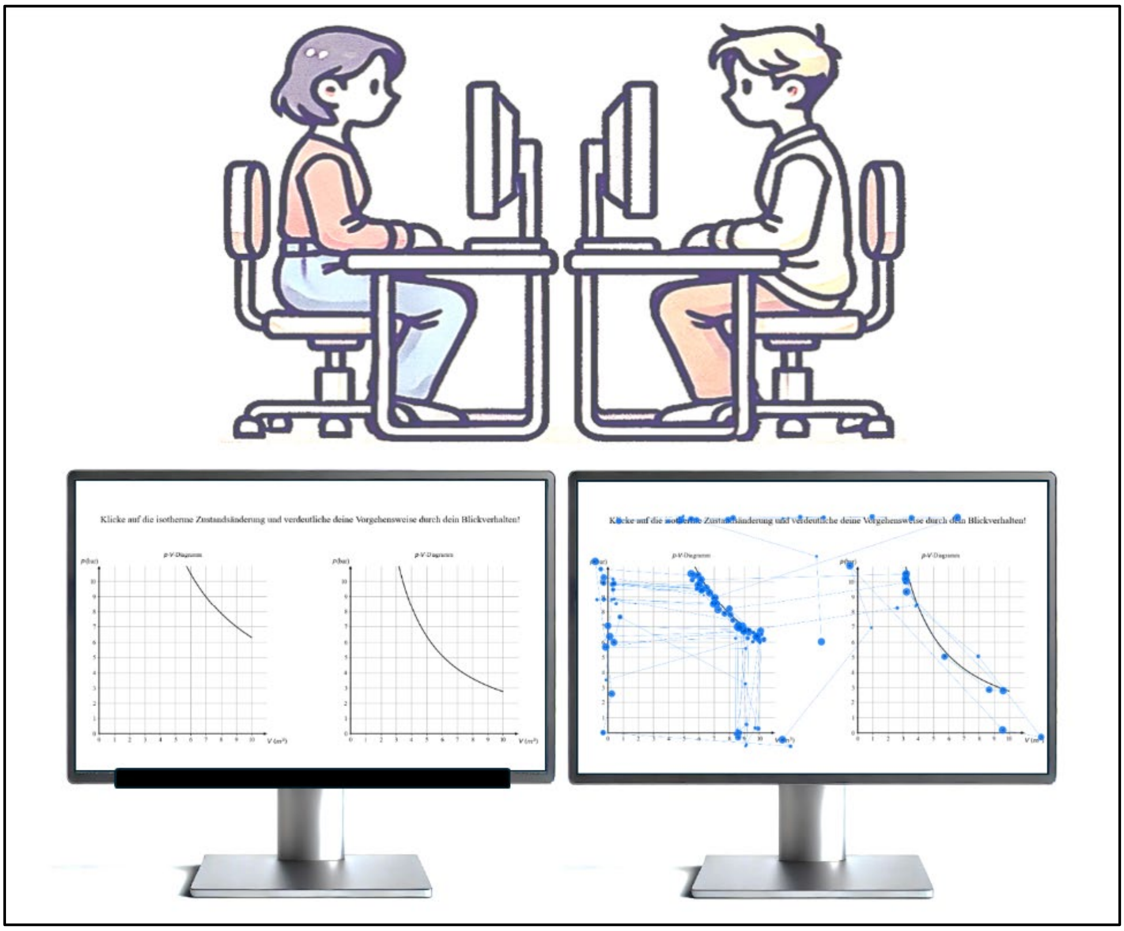

Fig. 3: Empirical experimental setup for investigating eye movement patterns during the identification of isothermal state changes in the p-V diagram. On the left, a test subject works on tasks to identify isothermal state changes while his eye movements are recorded using an eye tracker. On the right, the study director tracks the test subject's eye movements in real time. The screens show examples of the respective views of a test subject and the study director. The image was created using ChatGPT-4o. Item T1 from the empirical study published in PRPER [1].

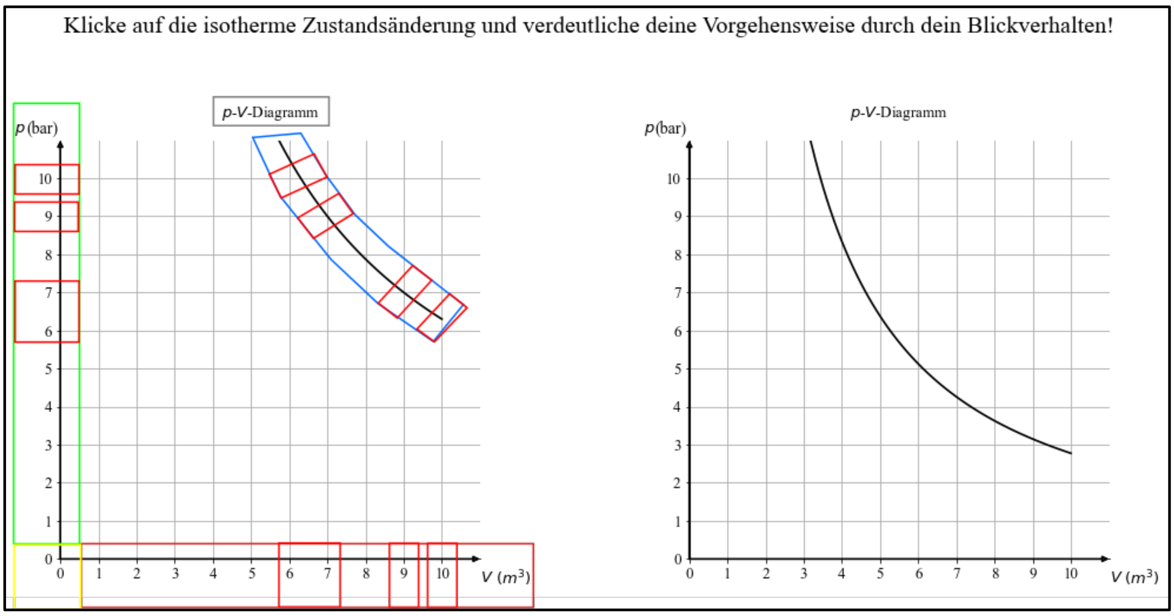

Fig. 4: Representation of areas of interest (AOIs) in the p-V diagram with isothermal state change for the investigation of strategies. Item T1 from the empirical study published in PRPER [1].

based on the AOIs were analyzed and a hierarchical cluster analysis (Ward method) was performed [12]. The verbal data was used to supplement this. For this purpose, the interviews were transcribed and subjected to a qualitative content analysis [13].

The relevant passages were then assigned to the clustered processing strategies.

In order to gain detailed insight into the gaze data of the processing strategies, the saccade directions of all participants were examined more closely, as these provide clues as to how the diagram is read and interpreted [8].

For this purpose, a kernel density estimation was applied to the saccade directions in order to represent them continuously.

8. Results

The heat map in Figure 5 already shows that the function graph, certain points on it, and the axes with the respective values were used in particular to solve the task.

Figure 6 shows examples of the gaze paths assigned to the classified processing strategies, as well as the results of the core density estimates of the saccade directions in polar diagrams. The cluster analysis identified two strategies: the 'focus on value pairs strategy' (see Fig. 6A) and the 'focus on function graph strategy' (see Fig. 6B).

In the "focus on value pairs" strategy, learners switch between points on the function graph and the axes (see Fig. 6A). The interviews reveal that they analyze two states of change by comparing the pressure and volume values. In doing so, they check whether the pressure of the first state corresponds to the volume of the second state and vice versa. One test subject explains:

"I compared the points to see if they changed by the same factor. And that was the case. That's why I decided to do it."

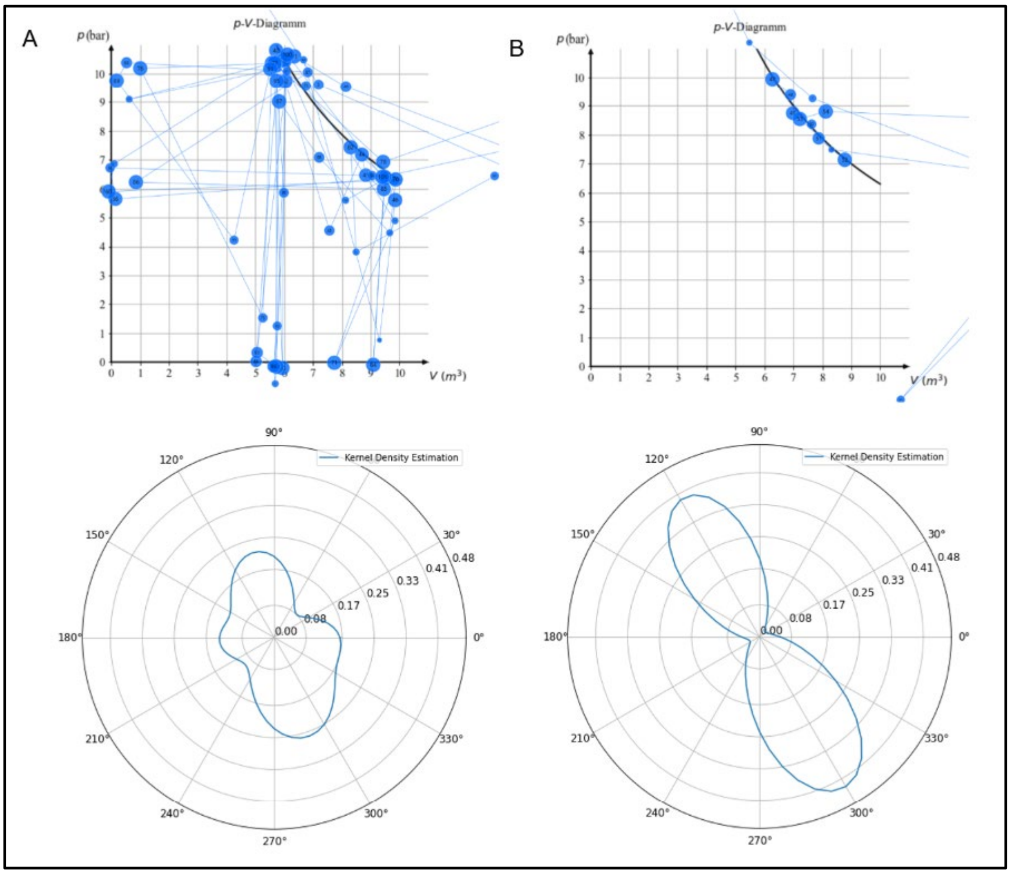

Fig. 6: Exemplary gaze paths for the processing strategies Focus on value pairs (A) and Focus on function graphs (B). The lower figures show the kernel density estimates (KDS) of the saccade directions for the respective processing strategies. Item T1 from the empirical study published in PRPER [1].

The eye movements are predominantly horizontal and vertical (see Fig. 6A, polar diagram).

The "focus on the function graph" strategy concentrates on the course of the function graph (see Fig. 6B).

According to interviews, learners primarily analyze the course and position of the curve to identify typical characteristics of isothermal state change. One test subject says, for example:

"... but it's just more curved than the other one. The other one is almost a straight line, well, almost. And that's why I chose the other one."

The saccade directions run diagonally (see Fig. 6B, polar diagram). The gaze patterns and verbal data reveal different approaches and difficulties.

9. Discussion

Clustering based on gaze data and analysis of saccade directions provide valuable insights into the processing strategies used by learners to identify isothermal state changes in the p-V diagram. In addition, the verbal data offer deeper insights into the processing strategies and reveal learning difficulties that hinder successful identification.

Two processing strategies were identified that are similar to those from the study by Thomaneck et al. [11]. The participants were evenly distributed across both strategies (50% each). On the one hand, the function graph was analyzed section by section using specific pairs of values, which is reflected in predominantly horizontal and vertical saccades. On the other hand, the graph was considered in its entirety, with diagonal saccades in particular indicating a holistic understanding of the function curve.

The analysis of the value pairs in particular represents a solid processing strategy that closely resembles a product equality test. Focusing on the function graph proves to be a successful strategy when learners check whether the graph shows the course of a right-angled hyperbola. However, this strategy for identifying isothermal state changes becomes prone to error when the focus is placed on the position of the graph, for example. Focusing exclusively on the position of the graph—for example, whether it runs "above" or "below" in the diagram—can lead to misinterpretations, as isothermal and adiabatic state changes can be similar in their graphical representation under certain conditions. The decisive difference, however, lies in the functional course of the curve, which cannot be determined solely by its position in the diagram. This points to difficulties learners have in dealing with graphs of indirectly proportional functions. Detailed information on the processing strategies can be found in the study by Fehlinger et al [1].

From a future-oriented perspective for adaptive support systems, it is a great advantage that the saccade directions exhibit specific characteristics in the respective processing strategies. These enable precise conclusions to be drawn about the learners' approach based on their eye movement patterns, allowing individually tailored assistance to be implemented in a targeted manner.

10. Conclusion

Previous studies have shown that learners often have problems analyzing function graphs, while teachers often fail to recognize these difficulties [2–4]. However, there are fundamentally suitable didactic approaches—such as the use of multiple forms of representation, the conscious switching between different representations, or the use of interactive visualization tools—that can specifically contribute to the promotion of these skills.

The analysis of eye movements and verbal data provides valuable insights into the processing strategies and specific learning difficulties of learners.

A key lesson learned is that learners often focus on irrelevant aspects when looking at graphs and therefore find it difficult to distinguish between isothermal and adiabatic processes. This highlights the need for targeted assistance to improve understanding of thermodynamic state changes.

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